CHIN. PHYS. LETT. Vol. 28, No. 6 (2011) 064501

A Pair Correlation Function Characterizing the Anisotropy of Force Networks *

SUN Qi-Cheng(孙其诚)[1**], JI Shun-Ying(季顺迎)[2]

[1] *State Key Laboratory of Hydroscience and Engineering, Tsinghua University, Beijing 100084*

[2] *State Key Laboratory of Structural Analysis for Industrial Equipment, Dalian University of Technology, Dalian 116023*

(Received 2 March 2011)

Force networks may underlie the constitutive relations among granular solids and granular flows and inter-state transitions. However, it is difficult to effectively describe the anisotropy of force networks. We propose a new pair correlation function $g(r, \theta)$ to describe the characteristic lengths and orientations of force chains that are composed of particles with contact forces greater than the threshold values. A formulation $g(r, \theta) \approx a(r) + b(r)\cos 2(\theta - \pi/2)$ is used to fit the $g(r, \theta)$ data. The characteristic lengths and orientations of force networks are then elucidated.

PACS: 45.70.−n, 81.05.Rm DOI: 10.1088/0256-307X/28/6/064501

In dense granular packings, multiple force transmission pathways may be established to support external loadings. Even small perturbations of loadings can sensitively induce the pathways. Elastic energy of granular assembly is stored in force chains; evolution of force chains results in one part of the elastic energy being transformed into kinetic energy, or vice versa, while the remainder of the elastic energy is dissipated. Therefore, structure and moment of such a force network are dominant in determining the major properties of granular materials, including elasticity, plasticity, failure and flowability [see Refs. [1,2] for granular solids and Refs. [3,4] for granular flows]. Although researchers have taken great efforts, precise understanding of the disordered spatial structure of force chains has remained elusive. For example, the authors of Refs. [5,6] found that the lengths of a force chain follow an exponential distribution. We also previously found that the probability density distribution of force chain lengths obeys a power law and remarkably, the exponent is not affected by packing fractions and inter-particle friction.[7] Complex networks offer a multitude of statistical measures for the quantitative characterization of evolving force networks, many of which can be calculated by manipulating an adjacency matrix.[8]

The pair correlation function $g(r)$ may be an effective tool in studying disordered force networks.[9,10] It was shown that the height of the first peak of $g(r)$, g_1, indicates signatures of both structured and jamming transitions. The $g(r)$ function focuses on the statistics of spatial locations of particles, while ignoring the more important orientations of the subsequently connected particles, that is, force chains. The anisotropy of structure can be characterized by a fabric tensor, which is defined by the distribution of contact normal vectors.[11,12] It incorporates measurements of both the magnitude of the force and the positions of particles in contact. Nevertheless, the fabric tensor cannot represent the spatial properties of force chains.

Since $g(r)$ provides an initiation point for such problems, we propose a variant pair correlation function $g(r, \theta)$. For a two-dimensional system, $g(r, \theta)$ is defined as

$$g(r, \theta) = \left(\frac{S}{N}\right)\frac{n(r, \theta)}{r dr d\theta}, \qquad (1)$$

where S is the area of the granular assembly, N is the total number of particles and $n(r, \theta)$ is the number of particles located within the area $r dr d\theta$. The parameter θ represents the orientation of the area with respect to the horizontal direction. Obviously, $g(r, \theta)$ represents the probability of finding two particles separated by a distance r in the direction of θ. In this work, $dr = 0.1\langle D\rangle$, $d\theta = \pi/180$. $\langle D\rangle$ is the mean diameter of particles. The proposed $g(r, \theta)$ is expected to analyze the anisotropy of a force network.

In this work, 10000 round disks were generated in a two-dimensional square cell. The particle diameters obey a Gaussian distribution between 0.025 m and 0.1195 m. The density is 2650 Kg$\cdot$m^{-3} and normal and tangential spring stiffness values both are 1×10^7 N/m. The inter-particle friction is assumed to be 0.1. Initially, the four walls move inwards to isotropically compress the assembly, leading to a well-jammed state with a fraction $\phi = 0.8411$. Subsequently, the left and right walls stop moving, but the top and bottom walls move further inwards to induce an anisotropic stress in the granular assembly, that is, the principal stress is along the vertical direction.

When an external force such as gravity or pressure is applied to a granular assembly, the propagation path of the force is a crucial consideration in

*Supported by the National Basic Research Program of China (2010CB731504), the National Natural Science Foundation of China (11034010) and the Research Fund of the State Key Laboratory of Hydroscience and Engineering, Tsinghua University (2008-ZY-6, 2010-TC-1)

**Email: qcsun@tsinghua.edu.cn

certain applications. A broad consensus has reached that the spatial distribution of contact forces characterizes strong concentration along force chains and these strong chains typically extend on space scales much larger than the grain dimensions, e.g. lengths of about 10 particle diameters. It is further found that only around 40% of contacts carry greater than the mean interparticle normal force $\langle f\rangle$, but hold 80% of the elastic energy in the system, which implies that strong force chains carry forces larger than the mean dominate granular properties.[7] Obviously, the threshold value for f_c is essential for defining a strong force chain. As shown in the inset of Fig. 1, the particles in a local area are depicted if the interparticle force is greater than f_c, where $f_c = 1.0\langle f\rangle$, $1.2\langle f\rangle$, $1.5\langle f\rangle$ and $1.8\langle f\rangle$, respectively. First, it can be observed that at larger f_c, fewer particles remain but the anisotropic configuration is more clearly displayed. The description of such a network with $g(r,\theta)$ is the main objective of this work. Second, we note that the value of the mean diameter of remaining particles increases as f_c increases, which indicates that strong forces are primarily transferred by large particles. It may be interpreted that large particles are easily jammed and act as backbones to support the assembly. The small particles experience smaller stresses because there remains a certain degree of agitation room between themselves and their neighbors. Therefore, the potential crushing of large particles may be more severe than that of smaller ones.

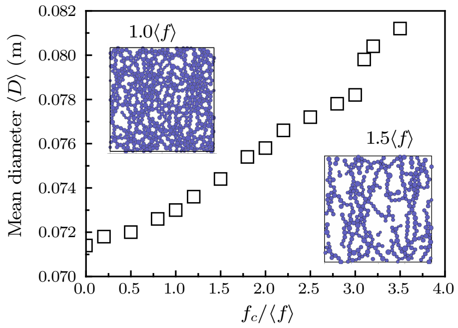

Fig. 1. The mean diameter of the particles in force chains. The interparticle force is greater than f_c. The insets are local networks at $f_c = 1.0\langle f\rangle$ and $1.5\langle f\rangle$.

The $g(r,\theta)$ distributions are calculated at different f_c. Taking $g(r,\theta)$ at $f_c = 1.5\langle f\rangle$ such as in Fig. 2, for each angle of θ, $g(r,\theta)$ possesses an oscillating shape characteristic of any disordered medium and clear peaks exist at approximate values of $r = \langle D\rangle$, $2\langle D\rangle$ and $3\langle D\rangle$. As $r\to\infty$, $g(r,\theta)$ approaches 1. Meanwhile, we find that at $r = \langle D\rangle$ and $r = 2\langle D\rangle$, two peaks of equal height exist at around $\theta = \pi/2$ and $3\pi/2$, which indicates the orientation of the principal stress in the anisotropically compressed assembly. Due to strong scattering at around $3\langle D\rangle$ of $g(r,\theta)$, we study the height of $g(r,\theta)$ at $r = \langle D\rangle$ and $2\langle D\rangle$ for different f_c.

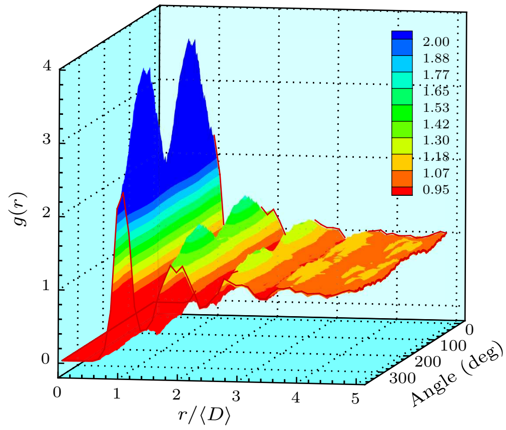

Fig. 2. Pair correlation function $g(r,\theta)$ at $f_c = 1.5\langle f\rangle$. The distance r is given in units of the mean particle diameter $\langle D\rangle$. Two clear peaks exist at $\pi/2$ and $3\pi/2$ at both $r = \langle D\rangle$ and $2\langle D\rangle$.

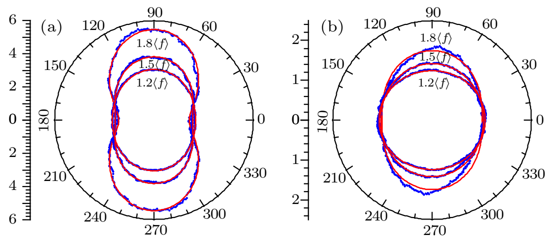

Fig. 3. Characteristics of $g(r,\theta)$ distribution at $r = \langle D\rangle$ (a) and $r = 2\langle D\rangle$ (b). Blue dots are calculations and solid curves are fittings. The corresponding f_c is indicated for individual curves.

In Fig. 3, $g(r,\theta)$ data are represented as blue dots for $f_c = 1.8\langle f\rangle$, $1.5\langle f\rangle$ and $1.2\langle f\rangle$, respectively. Here we adopt a similar description to the Fourier series as a representation of the symmetry property $g(r,\theta) = g(r,\theta)+\pi$ as follows:

$$g(r,\theta)\approx a(r)+b(r)\cos 2(\theta-\pi/2), \tag{2}$$

where $\pi/2$ indicates the principal external loadings perpendicular to the horizontal direction. It is interesting to note that the parameter b may limit the degree of anisotropy, that is, a larger value of b indicates higher anisotropy. The fittings are shown as solid curves, which provide significantly more smooth descriptions. The parameter b in the left figure for $\langle D\rangle$ is much greater than that in the right one for $2\langle D\rangle$. For $r = 3\langle D\rangle$ and even larger regions, b would become much smaller and the anisotropy may even disappear for longer ranges. Figure 4 shows the dependence of b on f_c and r. When $r = \langle D\rangle$, b is 1.663 at $f_c = 1.8\langle f\rangle$, while b is already close to 0 at $f_c\approx 0.8\langle f\rangle$, which implies an approximately uniform distribution of $g(r,\theta)$ at smaller values of f_c. At $r = \langle D\rangle$, the parameter b implies the distribution of normal vectors of interparticle contacts and it shares a similar meaning with the fabric tensor. More importantly, for $r \geq 2\langle D\rangle$, b represents the orientation distributions of force networks carrying forces greater than a threshold value.

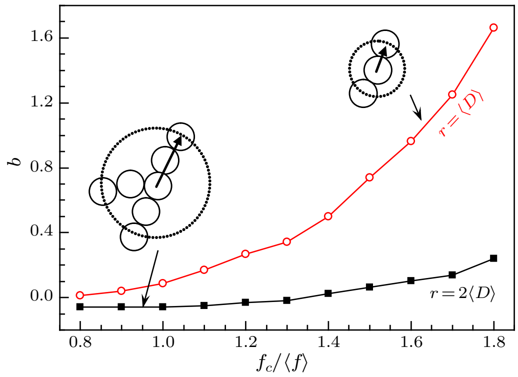

Fig. 4. Variation of parameter b in Eq. (2). A larger value of b implies higher anisotropy.

From Eq. (2), the characteristic length and orientation of a force network might be preliminarily determined. For example, at $f_c = 1.5\langle f\rangle$, $g(r,\theta)$ is clearly larger than 1 until $r = 3\langle D\rangle$, that is, four particles. We may say that the typical length of the force chain under the condition of $f_c = 1.5\langle f\rangle$ equals the combined diameters of four particles. Regarding the orientation, it is very clear that the peaks exist at $\theta = \pi/2$ and $\theta = 2\pi/2$ for $r \leq 3\langle D\rangle$, which indicates that the orientation of such a force network is normal to the horizontal direction, that is, parallel to the direction of the principal stress. As f_c increases, for example, $f_c = 1.8\langle f\rangle$, the typical length of a force chain is found to equal the combined diameters of three particles.

In summary, the primary characteristic of a force network is anisotropy, both intrinsic and induced. The quantification of such anisotropy remains difficult. Therefore, the new pair correlation function $g(r,\theta)$ proposed here is demonstrated to be an effective tool in studying the structural signatures of these force networks, in terms of both characteristic lengths and orientations.

References

[1] Jiang Y M and Liu M 2009 *Granular Matter* **11** 139
[2] Kuhn M R 1999 *Mech. Mater.* **31** 407
[3] Campbell C S 2006 *Powder Technol.* **162** 208
[4] Ji S Y and Shen H H 2008 *J. Rheol.* **52** 87
[5] Peters J F, Muthuswamy M, Wibowo J and Tordesillas A 2005 *Phys. Rev.* E **72** 041307
[6] Muthuswamy M and Tordesillas A 2006 *J. Stat. Mech.* P09003
[7] Sun Q, Jin F, Liu J and Zhang G 2010 *Int. J. Mod. Phys.* B **24** 5743
[8] Tordesillas A, O'Sullivan P and Paramitha D M W 2010 *Comptes Rendus Mecanique* **338** 556
[9] Donev A, Torquato S and Stillinger F H 2005 *Phys. Rev.* E **71** 011105
[10] Cheng X 2010 *Phys. Rev.* E **81** 031301
[11] Thornton C and Barnes D J 1986 *Acta Mech.* **64** 45
[12] Oda M, Nemat-Nasser S and Mehrabadi M M 1982 *Int. J. Numer. Anal. Methods Geomech.* **6** 77